(Defn):- Let E be a nls over field K of real or Complex nos. Then K it self is a nls over K w.r.t. the norm defined

$\|\alpha\| = |\alpha|$ for all $\alpha \in K$. A linear transformation, T from E into K is called a linear functional on E.

Corollary:- A linear function T on a nls E iff it is Continuous it is Continuous at the origin.

(Defn) Bounded linear transformations:- A linear transformation T from a nls E in a nls F is said to be bounded if there exist a +ve real number m such that $\|T(x)\| \leq m\|x\|$ for all $x \in E$.

A linear functional T is a nls E is said to be bounded if there exists $m>0$ such that -

$|T(x)| \leq m.\|x\|$ for all $x \in E$.

Theorem

(1) A linear transformation T from a nls E into a nls F is Continuous iff it is bounded.

or, QNo→ A linear transformation T from a normed linear space E into a normed linear space F is Continuous iff T is bounded in the sense that exists a +ive real number m such that

$\|T(x)\| \leq m \|x\|$ for all $x \in E$.

Proof:- Suppose that T is Continuous, If Possible let T be not bounded. Then, for every +ive real no. m, there exists a Point $x \in E$ Such that $||T(x)|| > m||x||$. In Particular, for every +ive integer n, there exists $x_n \in E$, Such that

$||T(x_n)|| > n||x_n||$, We define

$$y_n = \frac{x_n}{n||x_n||} \quad \text{Then } ||y_n|| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

$\therefore \{y_n\}$ Converges to 0. Now

$$||T(y_n)|| = \left|\left|T\left(\frac{x_n}{n||x_n||}\right)\right|\right| = \frac{1}{n||x_n||} \cdot ||T(x_n)|| > 1 \text{ for}$$

all n.

$\therefore \{T(y_n)\}$ ~~does not~~ doesn't Converge to $T(0) = 0$.

So T is not Continuous at 0, This is Contradiction. Hence T must be bounded.

Conversely, let T be bounded. Then there exists, $m > 0$ Such that

$||T(x)|| \leq m||x||$ for all $x \in E$.

Let $\epsilon > 0$ be given, We Choose $\delta = \frac{\epsilon}{m}$

Then, $||x|| < \delta \Rightarrow ||T(x)|| \leq m\delta = \epsilon$.

Therefore, T is Continuous at orgin and So, T is a Continuous.

(Def^n) **Bounded Set in a metric Space**:- Let (E, d) be a a metric Space A Subset A of E is Said to be bounded if there exists $a \in E$ and $K > 0$ Such that $d(x, a) \leq K$ for all $x \in A$.

i.e. $x \in S_k[a]$ for $x \in A$ i.e. $A \subseteq S_k[a]$. So, A is bounded if it is continuous in some closed sphere.
If E is a nls then a subset A of E is bounded if there exists $a \in E$ & $K > 0$ such that

$$\|x - a\| \leq K \text{ for all } x \in A.$$

i.e., if $x \in S_k[a]$ for all $x \in A$ or if $A \subseteq S_K[a]$.

Theorem
Q.No.→ A linear transformation T from a nls E into a nls F is continuous iff the image $T(S)$ of the closed unit ball (unit sphere) $S = \{x \in E : \|x\| \leq 1\}$ of E is a bounded in F.

Proof:- Let T be continuous. Then it is bounded. So there exists $m > 0$ such that $\|T(x)\| \leq m\|x\|$ for all $x \in E$. Now, $x \in S \Rightarrow \|x\| \leq 1$.

$\therefore \|T(x)\| \leq m$ i.e. $T(x) \in S_m[0]$

$\therefore T(S) \subseteq S_m[0]$. Hence $T(S)$ is a bounded set in F.

Conversely, Let $T(S)$ be bounded in F. Then there exists a closed ball $S_\epsilon[0]$ such that $T(S) \subseteq S_\epsilon[0]$.
Now, if $x = 0$ then clearly $\|T(x)\| \leq \epsilon\|x\|$.
If $x \neq 0$, we choose $y = \dfrac{x}{\|x\|}$ then $\|y\| = 1$ and so $y \in S$. Hence $\|T(y)\| \leq \epsilon$

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \epsilon$$

$$\text{or, } \frac{1}{\|x\|}\|T(x)\| \leq \epsilon$$

$$\text{or, } \|T(x)\| \leq \epsilon\|x\|$$

$\therefore$ T is bounded and so it is continuous.